Tribhuwan University

Institute of Science and Technology

2074

Bachelor Level / second-semester / Science — Full marks: 80

Computer Science and Information Technology(MTH163) — Pass marks: 32

(Mathematics II) — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all questions:

Group A (10 x 2 = 20)

1. What are the criteria for a rectangular matrix to be in echelon form?

2. prove that (a) $(A^T)^T = A$ (b) $(A + B)^T = A^T + B^T$, Where A and B denote matrices whose size are appropriate for the above mentioned operations.

3. Define square matrix. Can a square matrix with two identical columns be invertible? Why or why not?

4. Let A and B be two square matrices. By taking suitable examples, show that even though AB and BA may not be equal, it is always true that detAB = detBA.

5. Using Cramer's rule solve the following simultaneous equations:

$5x + 7y = 3$

$2x + 4y = 1$

6. Define vector space with suitable examples.

7. Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vector u and v such that W = Span {u, v}.

8. What are necessary and sufficient conditions for a matrix to be invertible?

9. Determine whether the pair of vectors $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ are orthogonal or not?

10. What do you understand by least square line? Illustrate.

Group B (5 x 4 = 20)

11. What are the criteria for a transformation T to be linear? If $T: \mathbf{R}^2 \to \mathbf{R}^2$ is defined by T(x) = 3x,

Show that T is a linear transformation. Also give a geometric description of the transformation

$$\mathbf{x} \mapsto A\mathbf{x}, \text{ where } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

12. Prove that if A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}.

13. Define subspace of a vector space V. Given v_1 and v_2 in a vector space V, let H = span $\{v_1, v_2\}$. Show that H is a subspace of V.

OR

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for a vector space V and x is in V, define the coordinate of x relative to the basis $\mathcal{B}$. Let

$$v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}, \mathcal{B} = \{v_1, v_2\}.$$

Then $\mathcal{B}$ is a basis for H = Span $\{v_1, v_2\}$. Determine is X is in H, and if it is, find the coordinate vector of x relative to $\mathcal{B}$

14. The mapping $T: P_2 \to P_2$ defined by $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ is a linear transformation.

a) Find the $\mathcal{B}$ matrix for T, when $\mathcal{B}$ is the basis $\{1, t, t^2\}$.

b) Verify that $[T(p)]_B = [T]_B[p]_B$ for each p in P_2.

15. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find a least square solution of Ax = b, and compute the associated least square error.

Group C (5 x 8 = 40)

16. Let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix for T. Then prove that: T map R^n on to R^m if and only if the columns of A span R^m; and T is one-to-one if and only if the columns of A are linearly independent. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map R^2 onto R^3?

17. Compute the multiplication of partitioned matrices for

$$A = \left[\begin{array}{ccc|cc} 2 & 3 & -1 & 0 & -2 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & 4 & 1 & 7 & -3 \end{array}\right]$$

and
$$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 3 \\ -3 & 7 \\ \hline -1 & 2 \\ 5 & 3 \end{array}\right]$$

18. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [X]_{\mathcal{B}}$ is one-to-one linear transformation from V into R^n. Let

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, b_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \text{ and } x = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}.$$

a) Show that the set $\mathcal{B} = \{b1, \ldots, bn\}$ is a basis of R^3.

b) Find the change of coordinates matrix for $\mathcal{B}$ to the standard basis.

c) Write the equation that relates x in R^3 to $[x]_{\mathcal{B}}$.

d) Find $[x]_{\mathcal{B}}$ for the x give above.

19. Diagonalize the matrix $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$ if possible.

OR

Suppose A = PDP^{-1}, where D is a diagonal n x n matrix. If $\mathcal{B}$ is the basis for R^n formed for the columns of P, then prove that D is the $\mathcal{B}$ matrix for the transformation. Define $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis $\mathcal{B}$ for R^2 with the property that the $\mathcal{B}$-matrix for T is a diagonal matrix.

20. What is a least squares solution? Find a least squares solution of Ax = b, where $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$.

OR

What do you understand by orthonormal set? Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of R^3, where

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}.$$

Prove that an m x n matrix U has orthonormal columns if and only if $U^TU = 1$.